Analysis I Lecture 22

Last time:

L'Hôpital rule:

If $\lim_{x \to x_0} f = \lim_{x \to x_0} g = 0$ or $\pm\infty$

and $\lim_{x \to x_0} \frac{f'}{g'}$ exists then $\lim_{x \to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f'}{g'}$

Taylor expansion:

(approximation)

f admits order n expansion at x_0 if:

$$f(x) = \overbrace{a_0 + a_1 (x-x_0) + \dots + a_n (x-x_0)^n}^{\text{polynomial part}} + \underbrace{\varepsilon_n(x)(x-x_0)^n}_{\text{Reminder}}$$

with $\lim_{x \to x_0} \varepsilon_n(x) = 0.$

Theorem If $f: I \to \mathbb{R}$ is C^{n+1} function and $x_0 \in I$ then f admits order n expansion at x_0 which is given by:

$$f(x) = \sum_{i=0}^{n} \underbrace{\frac{f^{(i)}(x_0)}{i!}}_{a_i} (x - x_0)^i + \underline{\underline{(x - x_0)^n \cdot \varepsilon_n(x)}}$$

Operations with order n expansion:

Let $f(x) = \overbrace{\sum_{i=0}^{n} a_i (x-x_0)^i}^{F(x)} + (x-x_0)^n \cdot \varepsilon_n$

$g(x) = \underbrace{\sum_{i=0}^{n} b_i (x-x_0)^i}_{G(x)} + (x-x_0)^n \cdot \varepsilon_n'$

Then **Addition:**

$$(f+g)(x) = \underbrace{F(x) + G(x)} + (x-x_0)^n \varepsilon'' =$$

$$= \sum_{i=0}^{n} \underbrace{(a_i + b_i)} (x-x_0)^i + (x-x_0)^n \varepsilon_n''$$

Multiplication:

$$f\cdot g(x) = \underbrace{F(x)\cdot G(x)}_{\text{only terms of } \deg \le n} + \varepsilon''\underbrace{(x-x_0)^n}_{} =$$

$$= \underbrace{\left(\sum_{i=0}^{n} a_i (x-x_0)^i\right)\left(\sum_{j=0}^{n} b_j (x-x_0)^j\right)}_{\text{collect terms of } \deg \le n}$$

$$+ \ \varepsilon''(x)\cdot(x-x_0)^n$$

Composition:

$$g \circ f(x) = g\big(f(x)\big) =$$

$$= \underbrace{G\big(F(x)\big)}_{\text{terms of deg} \le n} + \underline{\underline{\varepsilon''(x) \cdot (x - x_0)^n}} =$$

$$= \underbrace{\sum_{i=0}^{n} b_i \big(F(x) - x_0\big)^i}_{\text{terms of deg} \le n} + \varepsilon''(x) \cdot (x - x_0)^n$$

Example: Compute order 3 expansion of $\cos(\sin(x))$ at $x_0 = 0$.

Recall the expansions of $\cos(x)$ and $\sin(x)$:

$$\cos(x) = \overbrace{1 - \frac{x^2}{2}}^{F(x)} + \varepsilon(x) \cdot x^3$$

$$\sin(x) = \underbrace{x - \frac{x^3}{6}}_{G(x)} + \varepsilon'(x) \cdot x^3$$

$$\cos(\sin(x)) = \underbrace{F(G(x))}_{\text{terms of deg} \leq 3} + \varepsilon''(x)\cdot x^3 =$$

$$= 1 - \frac{\left(x - \frac{x^3}{6}\right)^2}{2} + \varepsilon''(x)\cdot x^3$$

$$= 1 - \frac{x^2 \underbrace{- 2\cdot x\cdot \frac{x^3}{6} + \frac{x^6}{36}}_{\text{deg} > 3}}{2} + \varepsilon''(x)\cdot x^3 =$$

$$= 1 - \frac{x^2}{2} + \varepsilon''(x)\cdot x^3$$

$\blacksquare$

<u>Example</u> Taylor expansion of e^x

at $x_0 = 1$:

$$e^x = \sum_{i=0}^{n} \frac{f^{(i)}(1)}{i!} \cdot (x-1)^i + \varepsilon_n(x) \cdot (x-1)^n$$

$$= \sum_{i=0}^{n} \frac{e^1}{i!} (x-1)^i + \varepsilon_n(x)(x-1)^n$$

$$= e \cdot \sum_{i=0}^{n} \frac{(x-1)^i}{i!} + \varepsilon_n (x-1)^n.$$

Taylor expansion and local extremums

<u>Theorem</u> 7.82 Let f be C^k function with k even. Let $f^{(i)}(x_0) = 0$ $\forall\, i < k$ then

- if $f^{(k)}(x_0) < 0$ then f has a local maximum at x_0
- if $f^{(k)}(x_0) > 0$ then f has a local minimum at x_0

Sketch of the proof:

Since f is C^k we Taylor expansion:

$$f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!} \cdot (x-x_0)^i + \varepsilon_k \cdot (x-x_0)^k$$

$$= \underline{f(x_0)} + 0 + \dots + 0 + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \varepsilon_k (x-x_0)^k$$

$$\Rightarrow \quad f(x) = f(x_0) + (x-x_0)^k \left(\frac{f^{(k)}(x_0)}{k!} + \varepsilon_k(x) \right)$$

Assume that $\varepsilon_k(x) = 0$

then $\dfrac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k$ has

local minimum at x_0 if $f^{(k)}(x_0) > 0$

local maximum at x_0 if $f^{(k)}(x_0) < 0$

since k is even.

In general $\varepsilon_k(x) \neq 0$ but we still have that $\lim_{x \to x_0} \varepsilon_k(x) = 0$

This enough to show the statement.

$\blacksquare$

Today:

Finish with differentiation:

- Convex functions
- Power series
- Taylor series

Convex and concave functions

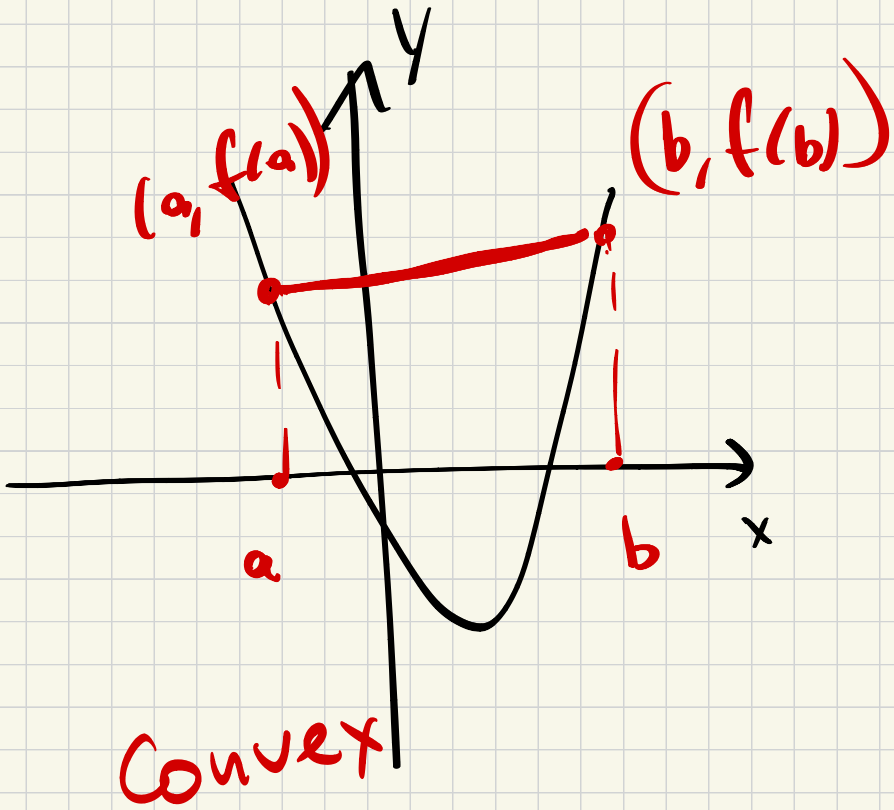
Definition ~7.88? We say that $f: I \to \mathbb{R}$

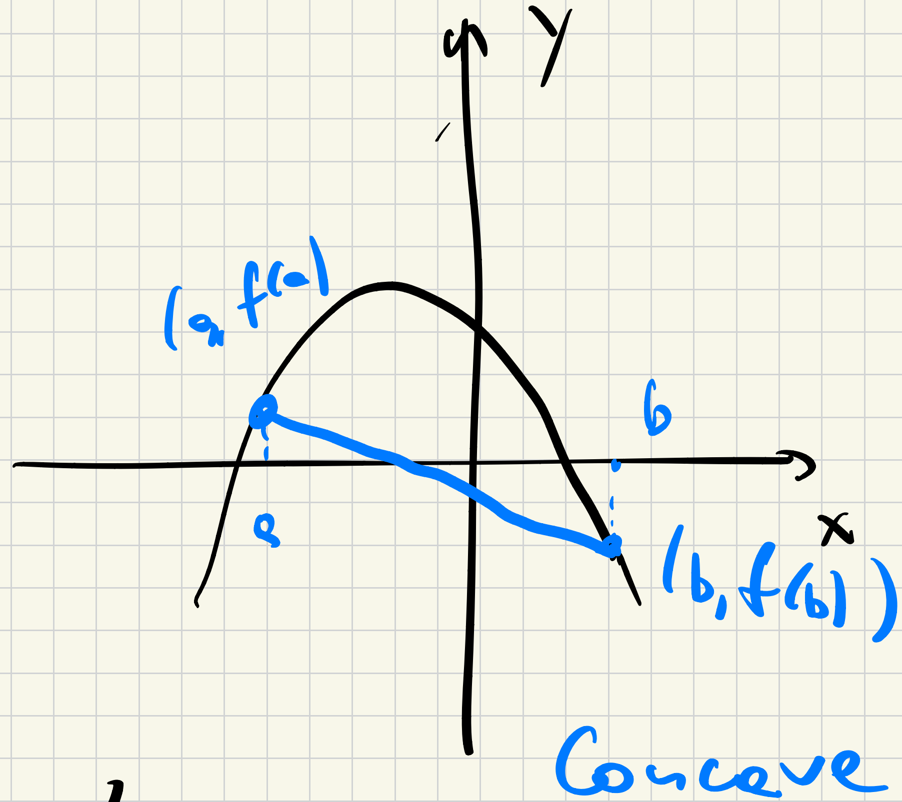
- is convex if $\forall a, b \in I$ we have

$$f(\lambda a + (1-\lambda) b) \leq \lambda f(a) + (1-\lambda) f(b) \quad \forall \lambda \in [0,1]$$

- is concave if $\forall a, b \in I$ we have

$$f(\lambda a + (1-\lambda) b) \geq \lambda f(a) + (1-\lambda) f(b) \quad \forall \lambda \in [0,1]$$


y
(a, f(a))
(b, f(b))
a
b
x
Convex


y
(a, f(a))
b
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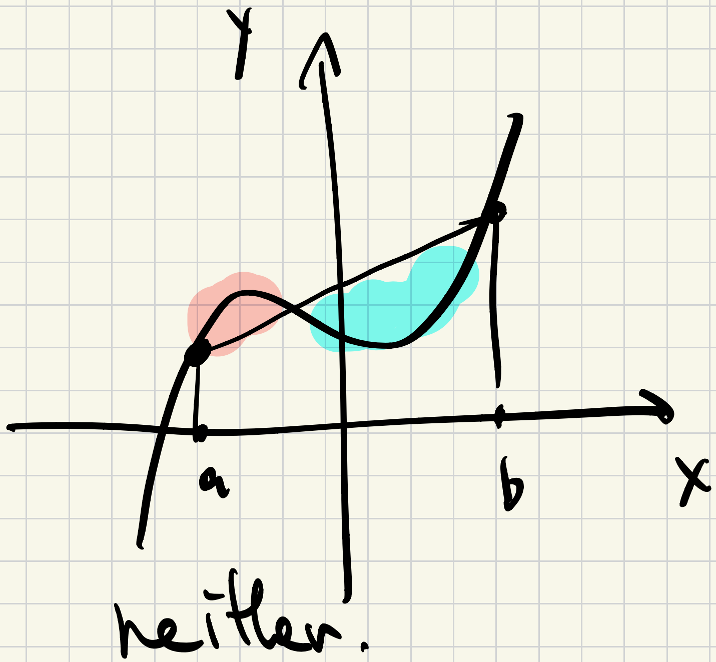
x
(b, f(b))
Concave


y
a
b
x
neither.

<u>Remark</u> If function f is convex then function $-f$ is concave and vice versa.

Theorem 7.90 Let $f: I \to \mathbb{R}$ be differentiable then

- f is convex if and only if f' is increasing
- f is concave if and only if f' is decreasing.

Corollary Let $f: I \to \mathbb{R}$ be C^2

then

f is convex if and only if $f''(x) \geq 0$

f is concave if and only if $f''(x) \leq 0$

Example $f(x) = x^2$ $f''(x) = 2 > 0 \Rightarrow$

$\Rightarrow f(x) = x^2$ convex on $\mathbb{R}$.

<u>Example</u> e^x , $\log(x)$

Both twice differentiable:

$(e^x)'' = e^x > 0$ for any $x \in \mathbb{R}$

$\Rightarrow$ e^x is convex

$$(\log(x))'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0 \quad \text{for } x \in \mathbb{R}_{>0}$$

so $\log(x)$ is concave.

Power series:

Definition 9.1 A power series centered at x_0 is an expression of the form:

$$\sum_{k=0}^{\infty} \underbrace{a_k}_{\text{coefficients}} (\overset{\text{variable}}{x} - \underbrace{x_0}_{\text{fixed point}})^k$$

where $a_k \in \mathbb{R}$.

We would like to treat power series as a function:

<u>E.g</u> evaluated at x_0 we get:

$$a_0 + \sum_{k=1}^{\infty} a_k (x_0 - x_0)^k =$$

$$= a_0 + 0 = a_0.$$

In general we can evaluate f if $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ converges

Definition (Domain of convergence).

For a power series $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ we define its convergence domain as

$$D = \left\{ x \in \mathbb{R} \;\middle|\; \sum_{k=0}^{\infty} a_k (x-x_0)^k \text{ is convergent} \right\}$$

Remark

- x_0 is always in the convergence domain.
- It could be the only point in the convergence domain.

Question: How to determine convergence domain?

Convergence domain is always an interval!

Theorem 9.4 Let $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ be a power series centered at $x_0 \in \mathbb{R}$.

then there is a number $R \geq 0$ or $+\infty$

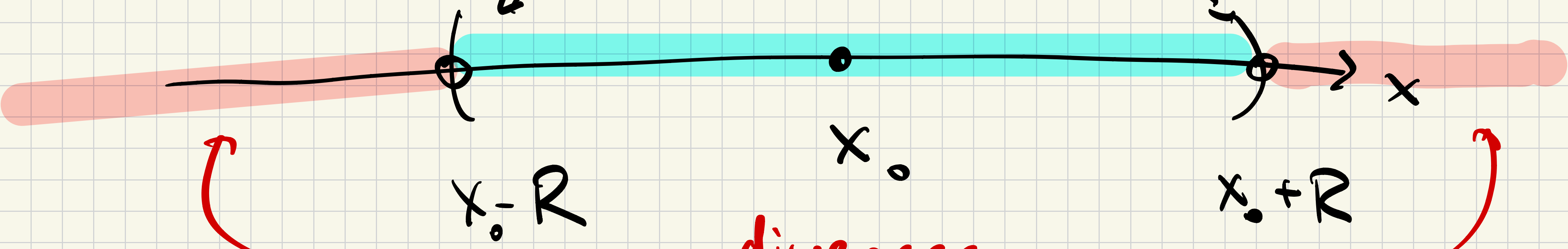
s.t. $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ converges if $|x-x_0| < R$

$\sum_{k=0}^{\infty} a_k (x-x_0)^k$ diverges if $|x-x_0| > R$

or $R<\infty$

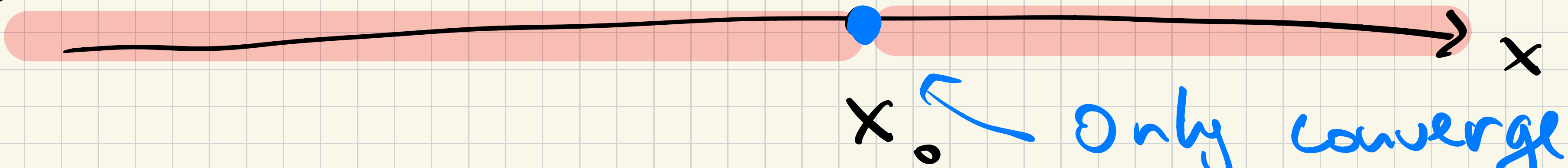
Still don't know

Converges



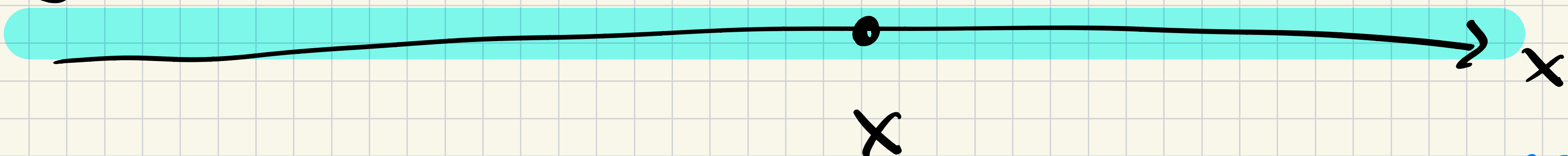
diverges

$R=0$



Only converge for $x=x_0$

$R=\infty$



Converge everywhere

This R is called

radius of convergence

for series $\sum_{k=0}^{\infty} a_k (x-x_0)^k$.

How to find radius of convergence?

Theorem 9.6 $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ power series with radius of convergence R.

- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \ell_1$ exists then $R = \frac{1}{\ell_1}$

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell_2$ exists then $R = \frac{1}{\ell_2}$.

Proof idea:

To apply ratio test to

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

$$\rightsquigarrow \lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| =$$

$$\lim_{n\to\infty} \underline{\underline{\left|\frac{a_{n+1}}{a_n}\right|}} \cdot \underbrace{|x-x_0|}$$

For series to converge we want this limit to be < 1

$$\Rightarrow \quad |x-x_0| < \frac{1}{\lim \left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\rho}$$

For second part

Apply root test for convergence

$\blacksquare$

Remark

Theorem 9.6 is also true for

$$\ell_i = \begin{cases} 0 \\ +\infty \end{cases} \quad \text{in that case}$$

we get

$$R = \begin{cases} +\infty \\ 0 \end{cases} \quad \left(= \text{"} \frac{1}{\ell_i} \text{"} \right)$$

Example Consider $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Here $a_n = \frac{1}{n!}$ so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 \implies R = +\infty$$

so series converges everywhere on $\mathbb{R}$.

Consider $\displaystyle\sum_{n=0}^{\infty} n x^n$

$a_n = n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$$\Rightarrow R = \frac{1}{1} = 1$$

So series converges for $x \in (-1, 1)$.
and diverges for $x \in (-\infty, -1) \cup (1, +\infty)$

What about $x = \pm 1$.

In our case if $x = 1$:

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=0}^{\infty} n \quad \text{which diverges}$$

And for $x = -1$ we get

$$\sum_{n=0}^{\infty} (-1)^n \cdot n \quad \text{which also diverges.}$$

$\blacksquare$

Proposition. Let $\sum_{k=0}^{\infty} a_k (x-x_0)^k$, $\sum_{k=0}^{\infty} b_k (x-x_0)^k$

be two power series with $a_i, b_i \geq 0$

radius of convergence R_1, R_2 resp.

Then radius of convergence of

$$\underbrace{\sum_{k=0}^{\infty} (a_k+b_k)(x-x_0)^k}_{\text{sum of power series.}} \quad \text{is} \quad \min(R_1, R_2)$$

Taylor series

Idea want to get expression of the form $f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$

Similar to Taylor expansion we can get this presentation when f is C^∞.

<u>Definition</u> Let $f: I \to \mathbb{R}$ be a C^∞-function then for $x_0 \in I$, the Taylor series of f at x_0 is:

$$T_{f, x_0}(x) = \sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(x_0)}{k!}}_{a_k} (x - x_0)^k$$

"Infinite Taylor expansion"

Next time:

When $f(x)$ is equal to $T_{f,x_0}(x)$?